CS5321 Numerical Optimization

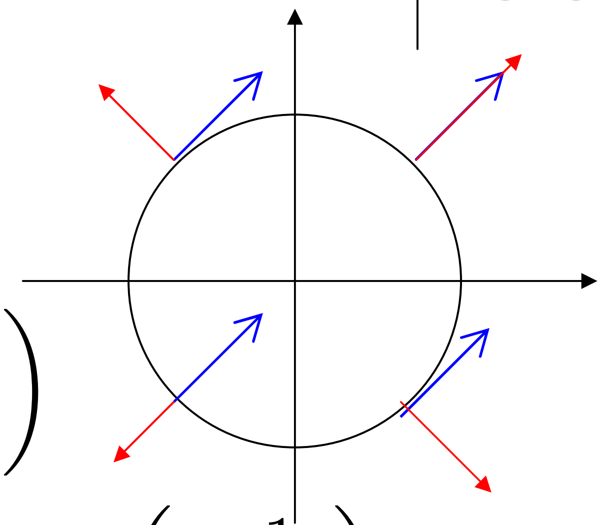
12 Theory of Constrained Optimization

General form

$$\min_{x\in\mathbf{R}^n} f(x) \text{ subject to } \begin{cases} c_i(x) = 0 & i \in \mathbf{E} \\ c_i(x) \geq 0 & i \in \mathbf{I} \end{cases}$$

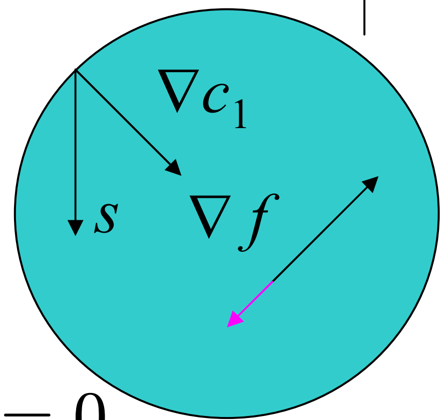
- **E**, **I** are index sets for equality and inequality constraints
- *Feasible set* $\Omega=\{x \mid c_i(x)=0,\ i\in\mathbf{E};\ c_i(x)\geq 0,\ i\in\mathbf{I}\}$
- Outline
  - Equality and inequality constraints
  - Lagrange multipliers
  - Linear independent constraint qualification
  - First/second order optimality conditions
  - Duality

A single equality constraint



- An example $\min_x f(x) = x_1 + x_2$
  subject to $c_1 = x_1^2 + x_2^2 = 2$
  $\nabla f(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \nabla c_1(x) = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}$
- The optimal solution is at $\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$
- Optimal condition: $\nabla f(x^*) = \lambda^* \nabla c_1(x^*)$
  - $\lambda^* = -1/2$ in the example

A single inequality constraint



- Example: $\min_x f(x) = x_1 + x_2$
  subject to $c_1 : 2 - x_1^2 - x_2^2 \geq 0$
- Case 1: the solution is inside c_1.
  - Unconstrained optimization $\nabla f(x^*) = 0$
- Case 2: the solution is on the boundary of c_1.
  - Equality constraint $\begin{cases} c_1(x^*) = 0 \\ \nabla f(x^*) = \lambda^* \nabla c_1(x^*) \end{cases}$
- Complementarity condition: $\lambda^* c_1(x^*) = 0$
  - Let $\lambda^*=0$ in case 1.

Lagrangian function

- Define $\mathcal{L}(x, \lambda) = f(x) - \lambda c_1(x)$
  - $\nabla_\lambda \mathcal{L}(x, \lambda) = -c_1(x)$
  - $\nabla_x \mathcal{L}(x, \lambda) = \nabla f(x) - \lambda \nabla c_1(x)$
- The optimality conditions of inequality constraint

$$\begin{cases} c_1(x^*) & = & 0 \\ \nabla f(x^*) & = & \lambda_1^* \nabla c_1(x^*) \end{cases}$$

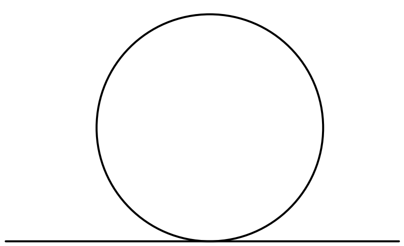
  which equal to $\nabla_\lambda \mathcal{L}(x, \lambda) = 0$ and $\nabla_x \mathcal{L}(x, \lambda) = 0$.
- $\mathcal{L}(x, \lambda)$ is called the Lagrangian function; λ is called the Lagrangian multiplier

Lagrangian multiplier

- At optimal solution x^*, $\nabla f(x^*) = \lambda_1 \nabla c_1(x^*)$
- If $c_1(.)$ changes δ unit, $f(.)$ changes $\approx \lambda_1 \delta$ unit.
  - λ_1 is the change of f per unit change of c_1.
  - λ_1 means the sensitivity of f to c_i.
  - λ_1 is called the *shadow price* or the *dual variable*.

Constraint qualification

- *Active set* $\mathbf{A}(x) = \mathbf{E} \cup \{i \in \mathbf{I} \mid c_i(x)=0\}$
- Linear independent constraint qualification (LICO)
  - The gradients of constraints in the active set $\{\nabla c_i(x) \mid i \in \mathbf{A}(x)\}$ are linearly independent
- A point is called regular if it satisfies LICO.
  - Other constraint qualifications may be used.



$$\begin{aligned} c_1(x) &= 1 - x_1^2 - (x_2 - 1)^2 \geq 0 \\ c_2(x) &= -x_2 \geq 0 \\ x &= (0,0) \end{aligned}$$

First order conditions

- A regular point that is a minimizer of the unconstrained problem must satisfies the KKT condition (Karush-Kuhn-Tucker)

$$
\begin{aligned}
\nabla_x L(x^*, \lambda^*) &= 0 & \\
c_i(x^*) &= 0 & \text{for all } i \in \mathbf{E} \\
c_i(x^*) &\geq 0 & \text{for all } i \in \mathbf{I} \\
\lambda^* &\geq 0 & \text{for all } i \in \mathbf{I} \\
\lambda^* c_i(x^*) &= 0 & \text{for all } i
\end{aligned}
$$

  - The last one is called *complementarity* condition

Second order conditions

- The *critical cone* $C(x^*, \lambda^*)$ is a set of vectors that

$$w \in C(x^*, \lambda^*) \Leftrightarrow \begin{cases} \nabla c_i(x^*)^T w = 0 & i \in \mathbf{E} \\ \nabla c_i(x^*)^T w = 0 & i \in \mathbf{I} \cap \mathbf{A}(x^*) \text{ with } \lambda_i^* > 0 \\ \nabla c_i(x^*)^T w \geq 0 & i \in \mathbf{I} \cap \mathbf{A}(x^*) \text{ with } \lambda_i^* = 0 \end{cases}$$

- Suppose x^* is a solution to a constrained optimization problem and λ^* satisfies KKT conditions. For all $w \in C(x^*, \lambda^*)$,

$$w^T \nabla_{xx}^2 L(x^*, \lambda^*) w \geq 0$$

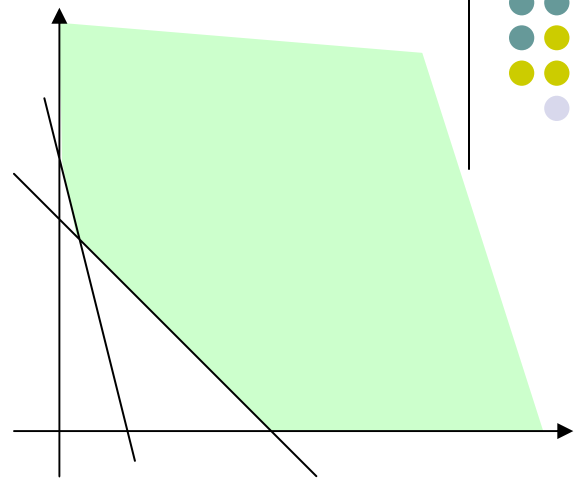
Projected Hessian

- Let $A(x^*)=[\nabla c_i(x^*)]_{i\in \mathbf{A}(x^*)}$: **A**(x*) is the active set.
- It can be shown that $C(x^*, \lambda^*)$ = **Null** $A(x^*)$
- **Null** $A(x^*)$ can be computed via QR decomposition

$$A(x^*)^T = Q\begin{pmatrix} R \\ 0 \end{pmatrix} = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix}\begin{pmatrix} R \\ 0 \end{pmatrix} = Q_1 R$$

  - span$\{Q_2\}$ = **Null** $A(x^*)$
- Define projected Hessian $H = Q_2^T \nabla_{xx}^2 L(x^*, \lambda^*) Q_2$
- The second order optimality condition is that H is positive semidefinite.

Duality: An example



$$\min_{x \geq 0} f(x) = 3x_1 + 4x_2$$

subject to

$$\begin{aligned} c_1 &: \quad x_1 + x_2 \geq 3 \\ c_2 &: \quad 2x_1 + 0.5x_2 \geq 2 \end{aligned}$$

- Find a lower bound for $f(x)$ via the constraints
  - Ex: $f(x) > 3c_1 \Rightarrow 3x_1 + 4x_2 > 3x_1 + 3x_2 \geq 9$
  - Ex: $f(x) > 2c_1 + 0.5c_2 \Rightarrow 3x_1 + 4x_2 > 3x_1 + 1.25x_2 \geq 7$
- What is the maximum lower bound for $f(x)$?

$$\max_{y \geq 0} g(y) = 3y_1 + 2y_2 \quad \text{s.t.} \quad \begin{aligned} y_1 + 2y_2 &\leq 3 \\ y_1 + 0.5y_2 &\leq 4 \end{aligned}$$

Dual problem

- Primal problem $\min_x f(x) \text{ s.t. } c_i(x) \geq 0$
  1. No equality constraints.
  2. Inequality constraints c_i are concave. ($-c_i$ are convex)
- Let $c(x)=[c_1(x), \ldots c_m(x)]^T$. The Lagrangian is $\mathcal{L}(x,\lambda) = f(x) - \lambda^T c(x),\ \lambda \in \mathbb{R}^m$.
- The dual object function is $q(\lambda)=\inf_x \mathcal{L}(x,\lambda)$
  - If $\mathcal{L}(:,\lambda)$ is convex, $\inf_x \mathcal{L}(x,\lambda)$ is at $\nabla_x \mathcal{L}(x,)=0$
- The dual problem is $\max_\lambda q(x) \text{ s.t. } \lambda \geq 0$
  - If $\mathcal{L}(:,\lambda)$ is convex, additional constraint is $\nabla_x \mathcal{L}(x,)=0$