CS3331 Numerical Methods

Quiz 9, Jan 6

Name: ________________________, ID: ________________________

1. Use $f(x) = a_1 x + a_0$ to approximate x^2.

   (a) What are a_1 and a_0 when using Taylor expansion to approximate x^2 at 0.5? (10pt)

   Let $g(x) = x^2$. Use Taylor expansion of $g(x)$ at 0.5.

   $$f(x) = g(0.5) + g'(0.5)(x - 0.5) = \frac{1}{4} + 2\frac{1}{2}(x - 0.5) = x - \frac{1}{4}$$

   $a_1 = 1$, $a_0 = -1/4$.

   (b) What are a_1 and a_0 when using minimum least square approximation in $[-1, 1]$? Which is $\min_{x\in[-1,1]} \|f(x) - x^2\|$ and use inner product $< f(x), g(x) >= \int_{-1}^{1} f(x)g(x)dx$. (10pt)

   $$\begin{pmatrix} \int_{-1}^{1} 1dx & \int_{-1}^{1} xdx \\ \int_{-1}^{1} xdx & \int_{-1}^{1} x \cdot xdx \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} \int_{-1}^{1} x^2 dx \\ \int_{-1}^{1} x \cdot x^2 dx \end{pmatrix}$$

   $$\begin{pmatrix} 2 & 0 \\ 0 & 2/3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 0 \end{pmatrix}$$

   $$a_1 = 0, a_0 = 1/3$$

2. The nth Chebyshev polynomial is defined as $T_n(x) = \cos(n \arccos(x))$.
Prove that $\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}} dx = 0$ for $m \neq n$. (10pt)
(You may need the formula $\cos(a)\cos(b) = \frac{1}{2}(\cos(a+b) + \cos(a-b))$)
(*No partial credits. Giving up or leaving it blank get 3 points.*)

Change variable: Let $\theta = \arccos(x) \Rightarrow x = \cos(\theta)$, and $dx = -\sin(\theta)$, $\sqrt{1-x^2} = \sqrt{1-\cos^2(\theta)} = \sin(\theta)$.

$$\begin{aligned}
\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}} dx &= \int_{-\pi}^{0} \frac{\cos(n\theta)\cos(m\theta)}{\sin(\theta)}(-\sin(\theta))d\theta \\
&= \frac{1}{2}\int_{0}^{\pi} [\cos((m+n)\theta) + \cos((m-n)\theta)]d\theta \\
&= \frac{1}{2}\left[\frac{1}{m+n}\sin((m+n)\theta) + \frac{1}{m-n}\sin((m-n)\theta)\right]\Big|_0^{\pi} = 0
\end{aligned}$$

3. What are 1-norm and infinity-norm of $x - x^2$ defined over $[-1, 1]$ (20pt)

1-norm : Let $f(x) = x - x^2 = x(1-x)$.
$f(x) \geq 0$ for $0 \leq x \leq 1$ and $f(x) < 0$ for $x < 0$ and $x > 1$.

$$\begin{aligned}
\|x - x^2\|_1 &= \int_{-1}^{1} |x - x^2| dx \\
&= \int_{-1}^{0} (x^2 - x)dx + \int_{0}^{1} (x - x^2)dx \\
&= \frac{1}{3}x^3 - \frac{1}{2}x^2\Big|_{-1}^{0} + \frac{1}{2}x^2 - \frac{1}{3}x^3\Big|_{0}^{1} \\
&= \frac{5}{6} + \frac{1}{6} = 1
\end{aligned}$$

inf-norm : $\|x - x^2\|_\infty = \sup_{-1 \leq x \leq 1} |x - x^2|$.
Let $f(x) = |x - x^2| = \left|\frac{1}{4} - \left(x - \frac{1}{2}\right)^2\right|$.
Maximal value of $f(x)$ can be only at three points: -1, $1/2$, and 1.

$$f(-1) = |-1-1| = 2, f(1/2) = 1/4, f(1) = 0.$$

So, $\|x - x^2\|_\infty = 2$.