CS3331 Numerical Methods

Quiz 5, Nov 28th

Name: __________________________, ID: __________________________

1. Let $\mathbf{A} = \begin{pmatrix} 4 & 9 & 1 & 3 \\ 0 & 8 & 0 & 5 \\ 0 & 0 & 7 & 2 \\ 0 & 6 & 0 & 10 \end{pmatrix}$ and $\mathbf{p} = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{pmatrix}$ be an approximation to $\mathbf{A}$'s eigenvector.

   (a) Compute the Rayleigh quotient of $\mathbf{p}$. (10pt)

   $\mathbf{p}^T\mathbf{p} = 1$. The Rayleigh quotient is

   $$\frac{\mathbf{p}^T\mathbf{A}\mathbf{p}}{\mathbf{p}^T\mathbf{p}} = \begin{pmatrix} 0.5 & 0.5 & 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} 4 & 9 & 1 & 3 \\ 0 & 8 & 0 & 5 \\ 0 & 0 & 7 & 2 \\ 0 & 6 & 0 & 10 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{pmatrix} = 55/4$$

   (b) Use the computed Rayleigh quotient, μ, as an eigenvalue approximation. Compute the 2-norm of the residual of (μ,$\mathbf{p}$). (10pt)
   The residual is $\mathbf{r} = \mathbf{A}\mathbf{p} - \mu\mathbf{p}$

   $$\begin{aligned} \mathbf{r} &= \begin{pmatrix} 4 & 9 & 1 & 3 \\ 0 & 8 & 0 & 5 \\ 0 & 0 & 7 & 2 \\ 0 & 6 & 0 & 10 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{pmatrix} - 55/4 \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{pmatrix} \\ &= \begin{pmatrix} 17/2 \\ 13/2 \\ 9/2 \\ 8 \end{pmatrix} - \begin{pmatrix} 55/8 \\ 55/8 \\ 55/8 \\ 55/8 \end{pmatrix} = \begin{pmatrix} 13/8 \\ -3/8 \\ -19/8 \\ 9/8 \end{pmatrix} \end{aligned}$$

   $\|\mathbf{r}\|_2 = \sqrt{13^2 + 3^2 + 19^2 + 9^2}/8 = \sqrt{620}/8$

2. Suppose all the eigenvalues of $\mathbf{A}$ are positive, and the largest one is 0.8. The convergent rate of the power method, applying to $\mathbf{A}$, is 0.75. What is the convergent rate of the shift-invert power method with shift=1? (10pt).

The second largest eigenvalue is 0.8*0.75=0.6.
In the shift-invert power method, the convergent rate is

$$\frac{|1-0.8|}{|1-0.6|} = 0.5$$

3. Apply one iteration of the QR method with shift $\mathbf{1}$ to $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.(20pt)

step 1. QR decompose $\mathbf{A} - \sigma\mathbf{I} = \mathbf{QR}$ (10pt)

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ 0 & 0 \end{pmatrix}$$

step 2. Assemble $\mathbf{RQ} + \sigma\mathbf{I}$ (10pt)

$$\begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$