CS 3331 Numerical Methods

Lecture 6: Iterative Methods for Solving Linear Systems

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Outline

- Background.

- Iterative methods
  - Jacobi method.
  - Gauss-Seidel method.
  - Successive over-relaxation.

Background

Motivation

- Some linear systems are large and sparse.
  - They cannot be stored into an $n \times n$ array.
  - Number of nonzero elements are $O(n)$.
  - The L-factor and U-factor are not sparse, but still large.
  - Time complexity of LU factorization is $O(n^3)$.
- Iterative methods
  - Access matrix elements constant times/iteration.
  - Need only $O(n)$ extra storage.
  - Transform $\mathbf{Ax} = \mathbf{b} \Rightarrow \mathbf{x}^k = \mathbf{C}\mathbf{x}^{k-1} + \mathbf{d}$.

Basic idea LVF pp.226

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= b_3 \\
a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 &= b_4
\end{aligned}
$$

Ex. Jacobi method

$$
\begin{array}{llllll}
x_1 = & & -\frac{a_{12}}{a_{11}}x_2 & -\frac{a_{13}}{a_{11}}x_3 & -\frac{a_{14}}{a_{11}}x_4 & +\frac{b_1}{a_{11}} \\
x_2 = & -\frac{a_{21}}{a_{22}}x_1 & & -\frac{a_{23}}{a_{22}}x_3 & -\frac{a_{24}}{a_{22}}x_4 & +\frac{b_2}{a_{22}} \\
x_3 = & -\frac{a_{31}}{a_{33}}x_2 & -\frac{a_{32}}{a_{33}}x_3 & & -\frac{a_{34}}{a_{33}}x_4 & +\frac{b_3}{a_{33}} \\
x_4 = & -\frac{a_{41}}{a_{44}}x_1 & -\frac{a_{42}}{a_{44}}x_3 & -\frac{a_{43}}{a_{44}}x_4 & & +\frac{b_4}{a_{44}}
\end{array}
$$

Fixed point iteration LVF pp.5,22,Last year's note

- Suppose $x = g(x)$ has a unique solution x^* in $[a, b]$. Then $x_{k+1} = g(x_k)$ will converge to x^* with any x_0, if

  1. $g(x)$ maps $[a, b]$ to $[a, b]$.
  2. $\exists \lambda < 1$ s.t. $|g(x) - g(y)| \leq \lambda |x - y|$ for all $x, y \in [a, b]$.

- Linear convergence with rate λ.
  $|x^* - x_{k+1}| = |g(x^*) - g(x_k)| \leq \lambda |x^* - x_k| \cdots \leq \lambda^k |x^* - x_0|$.

- In this chapter, $g(\mathbf{x}) = \mathbf{C}\mathbf{x} + \mathbf{d}$. $\|g(\mathbf{x}) - g(\mathbf{y})\| = \|\mathbf{C}\mathbf{x} - \mathbf{C}\mathbf{y}\|$

  $\|\mathbf{x}^* - \mathbf{x}_{k+1}\| = \|\mathbf{C}(\mathbf{x}^* - \mathbf{x}_k)\| = \cdots = \|\mathbf{C}^k(\mathbf{x}^* - \mathbf{x}_0)\| \leq \|\mathbf{C}^k\| \|\mathbf{x}^* - \mathbf{x}_0\|$

  - Converge condition: the maximum absolute value of the eigenvalues of $\mathbf{C}$ (spectral radius) is less than 1.

Iterative Methods

Jacobi method LVF pp.227

- Decompose $\mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U}$
  - $\mathbf{L}$ lower triangular part.
  - $\mathbf{D}$ diagonal part.
  - $\mathbf{U}$ upper triangular part.

$$\begin{aligned} \mathbf{D}\mathbf{x} &= (-\mathbf{L} - \mathbf{U})\mathbf{x} + \mathbf{b} \\ \mathbf{x} &= \underbrace{\mathbf{D}^{-1}(-\mathbf{L} - \mathbf{U})}_{\mathbf{C}}\mathbf{x} + \underbrace{\mathbf{D}^{-1}\mathbf{b}}_{\mathbf{d}} \end{aligned}$$

- If $\mathbf{A}$ is *strictly diagonal dominant*, then the maximum absolute valued eigenvalue of $\mathbf{C} < 1$.
  - Strictly diagonal dominant: $|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|$
- Operation count: number of iterations $\times$ time for matrix-vector multiplication.
- Easy to be parallelized.

Sensitive to equation order LVF pp.227,233

- Consider $\left\{ \begin{array}{rcl} 2x + y & = & 6 \\ x + 2y & = & 6 \end{array} \right.$

  – $\mathbf{C} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$

  – $\|\mathbf{C}\| = 1/2 < 1.$

- Exchange the equation order $\left\{ \begin{array}{rcl} x + 2y & = & 6 \\ 2x + y & = & 6 \end{array} \right.$

  – $\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$

  – $\|\mathbf{C}\| = 2 > 1.$

Gauss–Seidel method LVF pp.234

Consider a system $\left\{\begin{array}{rcrcrcr} 2x_1 & - & x_2 & + & x_3 & = & -1 \\ x_1 & + & 2x_2 & - & x_3 & = & 6 \\ x_1 & - & x_2 & + & 2x_3 & = & -3 \end{array}\right.$

Jacobi method

$$\begin{array}{rcrcrcrcr} x_1^{(k+1)} & = & & & 0.5x_2^{(k)} & - & 0.5x_3^{(k)} & - & 0.5 \\ x_2^{(k+1)} & = & -0.5x_1^{(k)} & & & - & 0.5x_3^{(k)} & + & 3 \\ x_3^{(k+1)} & = & -0.5x_1^{(k)} & + & 0.5x_2^{(k)} & & & - & 1.5 \end{array}$$

Gauss–Seidel method

$$\begin{array}{rcrcrcrcr} x_1^{(k+1)} & = & & & 0.5x_2^{(k)} & - & 0.5x_3^{(k)} & - & 0.5 \\ x_2^{(k+1)} & = & -0.5x_1^{(k+1)} & & & - & 0.5x_3^{(k)} & + & 3 \\ x_3^{(k+1)} & = & -0.5x_1^{(k+1)} & + & 0.5x_2^{(k+1)} & & & - & 1.5 \end{array}$$

Gauss–Seidel method LVF pp.237

$$
\begin{aligned}
\mathbf{x}^{(k+1)} &= -\mathbf{D}^{-1}\mathbf{L}\mathbf{x}^{(k+1)} - \mathbf{D}^{-1}\mathbf{U}\mathbf{x}^{(k)} + \mathbf{D}^{-1}\mathbf{b} \\
\mathbf{D}^{-1}(\mathbf{D}+\mathbf{L})\mathbf{x}^{(k+1)} &= -\mathbf{D}^{-1}\mathbf{U}\mathbf{x}^{(k)} + \mathbf{D}^{-1}\mathbf{b} \\
(\mathbf{D}+\mathbf{L})\mathbf{x}^{(k+1)} &= -\mathbf{U}\mathbf{x}^{(k)} + \mathbf{b} \\
\mathbf{x}^{(k+1)} &= -(\mathbf{D}+\mathbf{L})^{-1}\mathbf{U}\mathbf{x}^{(k)} + (\mathbf{D}+\mathbf{L})^{-1}\mathbf{b}
\end{aligned}
$$

- If $\mathbf{A}$ is symmetric positive definite, Gauss–Seidel converges with any initial guess.
- Like Jacobi method, it can diverge. But when converging, it is more faster than Jacobi method.
- Hard to parallelize.

Successive over-relaxation LVF pp.238

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{aligned}
$$

Linear combination of $\mathbf{x}^{(k)}$ and the vector from Gauss–Seidel.

$$
\begin{aligned}
x_1^{(k+1)} &= (1-\omega)x_1^{(k)} + \tfrac{\omega}{a_{11}}(b_1 \qquad\qquad\qquad\qquad - a_{12}x_2^{(k)} - a_{13}x_2^{(k)}) \\
x_2^{(k+1)} &= (1-\omega)x_2^{(k)} + \tfrac{\omega}{a_{22}}(b_2 - a_{21}x_2^{(k+1)} \qquad\qquad\qquad - a_{23}x_2^{(k)}) \\
x_3^{(k+1)} &= (1-\omega)x_3^{(k)} + \tfrac{\omega}{a_{33}}(b_3 - a_{31}x_2^{(k+1)} - a_{32}x_2^{(k+1)} \qquad\qquad\quad)
\end{aligned}
$$

- For $0 < \omega < 1$, the method is called *successive under-relaxation*.
- For $1 < \omega < 2$, the method is called *successive over-relaxation*.

Successive over-relaxation LVF pp.240

$$\begin{aligned}
\mathbf{x}^{(k+1)} &= (1-\omega)\mathbf{x}^{(k)} + \omega[-\mathbf{D}^{-1}\mathbf{L}\mathbf{x}^{(k+1)} - \mathbf{D}^{-1}\mathbf{U}\mathbf{x}^{(k)} + \mathbf{D}^{-1}\mathbf{b}] \\
\mathbf{D}^{-1}(\mathbf{D}+\omega\mathbf{L})\mathbf{x}^{(k+1)} &= [(1-\omega)\mathbf{I} - \omega\mathbf{D}^{-1}\mathbf{U}]\mathbf{x}^{(k)} + \omega\mathbf{D}^{-1}\mathbf{b} \\
(\mathbf{D}+\omega\mathbf{L})\mathbf{x}^{(k+1)} &= [(1-\omega)\mathbf{D} - \omega\mathbf{U}]\mathbf{x}^{(k)} + \omega\mathbf{b} \\
\mathbf{x}^{(k+1)} &= (\mathbf{D}+\omega\mathbf{L})^{-1}\left[((1-\omega)\mathbf{D} - \omega\mathbf{U})\mathbf{x}^{(k)} + \omega\mathbf{b}\right]
\end{aligned}$$

- $\mathbf{C} = (\mathbf{D}+\omega\mathbf{L})^{-1}((1-\omega)\mathbf{D} - \omega\mathbf{U})$.
- Determinant of $\mathbf{C}$ is $(1-\omega)^n$.

$$\det\mathbf{C} = \frac{\det((1-\omega)\mathbf{D} - \omega\mathbf{U})}{\det(\mathbf{D}+\omega\mathbf{L})} = \frac{(1-\omega)^n \prod_{i=1}^n d_i}{\prod_{i=1}^n d_i} = (1-\omega)^n$$

  – Always choose $0 < \omega < 2$.