CS3331 Numerical Method

Homework 9 and its solution

$f(x) = e^x - x$ is defined in $[-1, 1]$.

1. Compute $\|f(x)\|_1$

   $f(x) > 0$ in the interval $[-1, 1]$.

$$\begin{aligned} \|f(x)\|_1 &= \int_{-1}^{1} |f(x)| dx \\ &= \int_{-1}^{1} (e^x - x) dx \\ &= e^x - \frac{1}{2}x^2|_{-1}^{1} \\ &= e - e^{-1} \end{aligned}$$

2. Compute $\|f(x)\|_\infty$

$$\|f(x)\|_\infty = \sup_{x \in [-1,1]} |f(x)|$$

   $f'(x) = e^x - 1 = 0$. The extreme value of $f(x)$ is at $x = 0$. But it is the minimum value. The largest value of $f(x)$ in $[-1, 1]$ is at boundaries, either at -1 or 1.

$$f(1) = e - 1 \approx 1.718 > f(-1) = e^{-1} + 1 \approx 1.36$$

$$\|f(x)\|_\infty = f(1) = e - 1$$

3. Approximate $f(x)$ by Legendre polynomials of degree 0 and 1.

Legendre polynomials of degree 0 and 1 are 1 and x.

$$\begin{aligned} < f(x), 1 > &= \int_{-1}^{1} (e^x - x) dx \\ &= e^x - \frac{1}{2} x^2 |_{-1}^{1} = e - e^{-1} \\ < f(x), x > &= \int_{-1}^{1} (e^x - x) x dx \\ &= e^x x|_{-1}^{1} - \int_{-1}^{1} e^x dx - \frac{1}{3} x^3 |_{-1}^{1} \\ &= (e + e^{-1}) - (e - e^{-1}) - 2/3 = 2e^{-1} - 2/3 \end{aligned}$$

Approximation is

$$\begin{aligned} p(x) &= \frac{< f(x), 1 >}{< 1, 1 >} 1 + \frac{< f(x), x >}{< x, x >} x \\ &= \frac{e - e^{-1}}{2} 1 + \frac{2e^{-1} - 2/3}{2/3} x \end{aligned}$$

4. Approximate $f(x)$ by Chevyshev polynomials of degree 0 and 1.

Chevyshev polynomials of degree 0 and 1 are 1 and x.

$$\begin{aligned} < f(x), 1 > &= \int_{-1}^{1} \frac{e^x - x}{\sqrt{1 - x^2}} dx \\ < f(x), x > &= \int_{-1}^{1} \frac{(e^x - x)x}{\sqrt{1 - x^2}} dx \end{aligned}$$

We need to change variables.
Let $x = \cos\theta$. Then $dx = -\sin\theta d\theta$ and $\sqrt{1 - x^2} = \sin\theta$.

$$\begin{aligned} < f(x), 1 > &= \int_{0}^{\pi} (e^{\cos\theta} - \cos\theta) d\theta = \int_{0}^{\pi} e^{\cos\theta} d\theta \\ < f(x), x > &= \int_{0}^{\pi} (e^{\cos\theta} - \cos\theta) \cos\theta d\theta \end{aligned}$$

The integration is difficult. We just write the approximate as

$$p(x) = \frac{< f(x), 1 >}{< 1, 1 >} 1 + \frac{< f(x), x >}{< x, x >} x$$